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PIECEWISE-HOMOGENEOUS PLATES OF EXTREMAL STIFFNESS*

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The shapes of a finite number of foreign inclusions in an elastic plane that minimize its potential strain energy for a uniform stress field given at infinity are sought. The area of the inclusions is considered known and their contact with the fundamental plate material (matrix) is ideal. It is shown that for a state close to multilateral compression the boundary of each inclusion is optimal if the force interaction of the materials along it reduces to a constant pressure of definite intensity. The stress field in the inclusions here turns out to be uniform while the desired boundaries are equally-strong. Their actual determination reduces to solving well-studied modification of the conjugate boundary value problem (explicit in a number of cases) for analytic functions.

1. Let Γ be a set of m smooth closed curves Γ_k, $k = 1, 2, \ldots, m$, located arbitrarily in the exterior of each other in the E plane of the complex variable $z = x + iy$, S_k is a simply-connected domain of area q_k within Γ_k, S_0 is a multiconnected infinite domain supplementing $S_- = \bigcup S_k$ up to E, and n, t are the directions of a local curvilinear coordinate system on Γ along the normal and the tangent at any point ξ.

Each of the listed $m + 1$ domains is occupied by its own homogeneous and isotropic linearly elastic material of identical small thickness h with shear modulus μ_j and Poisson's ratio ν_j, $j = 0, 1, \ldots, m$, together with which the constant $\varkappa_j = (3 - \nu_j)/(1 + \nu_j)$ corresponding to a generalized plane stress state /1/ is also introduced. To be specific, we will assume $\mu_0 \geqslant \mu_k$, $k = 1, 2, \ldots, m$.

The loading conditions are given by components of the stress tensor T_∞ at infinity

$$\sigma_\alpha^\infty = P_0, \quad \sigma_y^\infty = Q_0, \quad \tau_{xy}^\infty = 0 \tag{1.1}$$

Let us consider the problem of minimizing the strain energy of a plate by selecting the optimal shape of the contours Γ_k. We have in the notation used

$$U(\varkappa_j, \mu_j, \Gamma_k, q_k, P_0, Q_0) \to \min_{\{\Gamma\}} = U_0 \tag{1.2}$$

The energy density $w(z)$ at each point of the plate can be represented in terms of the invariants I_1, I_2 of the stress tensor $T(z)$ /1/

$$8\mu_j(1 + \nu_j)\, w(z) = h[I_1^2(z) + 2(1 + \nu_j) I_2(z)] \tag{1.3}$$
$$z \in S_j, \quad j = 0, 1, 2, \ldots, m$$

The curves Γ_k are lines of discontinuity of $w(z)$. For functions of this kind, their limit values on Γ from S_0 and S_- will be denoted by the superscripts plus and minus, respectively.

The following asymptotic form results from (1.1) as $|z| \to \infty$:

$$I_1(z) = a_0 + O(|z|^{-2}), \quad a_0 = P_0 + Q_0$$
$$4I_2(z) = b_0^2 - a_0^2 + O(|z|^{-4}), \quad b_0 = Q_0 - P_0$$

$$16\mu_0(1 + \nu_0)\, w(z) = \text{ch} + O(|z|^{-4})$$
$$c = (1 - \nu_0) a_0^2 + (1 + \nu_0) b_0^2$$

Taking this into account, we can write the regularized functional as a convergent integral in E

$$U = \int_{S_0} [w(x, y) - c]\, dx\, dy + \int_{S_-} w(x, y)\, dx\, dy \tag{1.4}$$

The first term in (1.4) corresponds to a perturbation of the uniform field (1.1) in S_0

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induced by the inclusions, while the second is their intrinsic strain. For given forces P_0, Q_0 the minimum of U obviously corresponds to a plate of maximum stiffness.

The components of the tensor $T(z)$ are determined in each domain in terms of a pair of holomorphic potentials therein $\varphi(z), \psi(z)$, connected on Γ by continuity boundary conditions for the displacement vector $(u_0(z), v_0(z))$ and the normal forces $\sigma_n(\xi), \tau_{nt}(\xi)$:

$$\mu_k[\varkappa_0\varphi_0(\xi) - \xi\overline{\varphi_0'(\xi)} - \overline{\psi_0(\xi)}] = \mu_0[\varkappa_k\varphi_k(\xi) - \xi\overline{\varphi_k'(\xi)} - \overline{\psi_k(\xi)}] \tag{1.5}$$

$$\varphi_0(\xi) + \xi\overline{\varphi_0'(\xi)} + \overline{\psi_0(\xi)} = \varphi_k(\xi) + \xi\overline{\varphi_k'(\xi)} + \overline{\psi_k(\xi)} = f_k(\xi) \tag{1.6}$$

$$\xi \in \Gamma_k, \quad k = 1, 2, \ldots, m$$

$$4\varphi_0(z) = a_0 + O(|z|^{-1}), \quad 2\psi_0(z) = b_0 + O(|z|^{-1}), \quad |z| \to \infty \tag{1.7}$$

$$I_1(z) = 4\,\mathrm{Re}\,\varphi_j'(z), \quad I_1^2(z) + 4I_2(z) = 4\,|\bar{z}\varphi_j''(z) + \psi_j'(z)|^2, \quad z \in S_j, \quad j = 0, 1, 2, \ldots, m \tag{1.8}$$

The functions $f_k(z)$ defining the contact forces between the matrix and the inclusions are written down explicitly for convenience.

Let us pose problem (1.1) as a variational problem regarding the stationary value of the quantity U in the form (1.4) for moving contours Γ_k and isoperimetric constraints on the area of the domain S_k ($dl = |d\xi|$ is the differential of the contour arclength):

$$\int_{\Gamma_k}\left(x\frac{\partial x}{\partial n} + y\frac{\partial y}{\partial n}\right)dl = \int_{\Gamma_k}\left(x\frac{\partial y}{\partial l} - y\frac{\partial x}{\partial l}\right)dl = 2q_k \tag{1.9}$$

Equilibrium equations of the medium in the domains S_j and the stationarity condition on Γ are obtained in the form of relationships for the jump in the function $w(z)$ by using ordinary variational techniques /2, 3/

$$w^+(\xi) - w^-(\xi) = \lambda_k, \quad t \in \Gamma_k, \quad k = 1, 2, \ldots, m \tag{1.10}$$

It is known to be conserved if the shape of the contours Γ_k is such that the force interaction of the phases along each of them reduces to just normal pressure of a constant magnitude p_k and, this means to the accuracy of non-essential components $f_k(\xi) = p_k\xi$. The a priori assignment of such forces affords the possibility of solving problem (1.6) separately in the interior domains for contours of any kind: $2\varphi_k(z) = p_k z$, $\psi_k(z) = 0$, $z \in S_k$, $k = 1, 2, \ldots, m$. The stress field therein is here uniform according to (1.8):

$$I_1(z) = \text{const} = 2p_k, \quad I_2(z) = \text{const} = -p_k^2, \quad z \in S_k \tag{1.11}$$

and in particular $w^-(\xi) = \text{const}$. Then it follows from (1.10) that $w^+(\xi) = \text{const}$. When taking account of the representation $I_1(\xi) = \sigma_n + \sigma_t$, $I_2(\xi) = \tau_{nt}^2 - \sigma_n\sigma_t$ in the (n, t) coordinate system on Γ and also the conditions $\sigma_n^+ = p_k$, $\tau_{nt}^+ = 0$ this is equivalent to the well-known equal-strength requirment /2/ for the desired contours: $\sigma_t^+ = \text{const}$, which was earlier displayed in a number of papers as the necessary optimality condition in a plate with holes but not with elastic inclusions according to the Mises local plasticity criterion /1/. It is obtained in /3/ by a variational method for the energy criterion.

Now problem (1.6) for a domain S_0 under a given load on the eqally-strong interfaces of the media is identical with that considered in /4/ where it is shown in particular that $4\varphi_0 = a_0 z$, $z \in S_0$, $\sigma_t^+ = a_0 - \sigma_n^+ = a_0 + p_k$, while the boundary condition

$$2\psi_0(\xi) = (2p_k - a_0)\,\bar{\xi}, \quad \xi \in \Gamma_k \tag{1.12}$$

follows from (1.6) for $\psi_0(z) = b_0 z + \Omega_0(z)$.

On the other hand, the elimination of $\psi_0(\xi)$ from (1.5) and (1.6) after reduction yields on the contour Γ_k

$$p_k = -\sigma_n^- = \frac{\mu_k(\varkappa_0 + 1)a_0}{2\mu_0(\varkappa_k - 1) + 4\mu_k}, \quad \sigma_t^- = \sigma_n^- \tag{1.13}$$

It is characteristic that the magnitude of the pressure depends only on the global part of the tensor T_∞.

We will now find the value U_0 for the equally-strong boundaries. It follows from (1.11) that the energy density $w(z)$ in the interior domains S_k is constant while it results in S_0 in the form

$$w(x, y) - c \sim |\psi_0'(z)|^2 - {}^1/_4 b_0^2 = \mathrm{grad}^2\,\rho_1(z) - {}^1/_4 b_0^2$$

when taking account of the previous reasoning and the Cauchy-Riemann relations for the functions $\rho_1(z) = \mathrm{Re}\,\psi_0(z)$ and $\rho_2(z) = \mathrm{Im}\,\psi_0(z)$.

Consequently, the first component in (1.4) is converted into a curvilinear integral of

$\rho_1(\xi)\,\partial\rho_1(\xi)/\partial l$ in Γ, on the basis of Green's formula and the asymptotic form (1.7). By virtue of the identities $2\rho_1(\xi) = (2p_k - a_0)x$, $\partial\rho_1(\xi)/\partial l = (2p_k - a_0)\,\partial y/\partial l$, $x + iy = \xi \in \Gamma_k$, that follows from (1.12), it is reduced to (1.9) and consequently is proportional to the sum of the areas of all the inclusions.

We consequently obtain

$$U = \frac{h}{4}\sum_{k=1}^{m} \frac{(a_0 - 2p_k)^2 + b_0^2}{4\mu_0} q_k + \frac{(1-\nu_k)\,p_k^2}{\mu_k(1+\nu_k)} q_k$$

Substitution of the expression (1.13) for p_k finally yields

$$U = \frac{h}{32\mu_0}\sum_{k=1}^{m} \times$$
$$\frac{2a_0^2\{[\mu_0(\varkappa_k - 1) - \mu_k(\varkappa_0 - 1)]^2 + \mu_0\mu_k(\varkappa_0 + 1)(\varkappa_k - 1)^2\} + b_0^2}{[\mu_0(\varkappa_k - 1) + 2\mu_k]^2} q_k$$

2. The existence of equally-strong contours is related to the solvability of condition (1.12) for the function $\psi_0(z)$ a unit potential explicitly dependent on the shape of Γ (in terms of the relationship between ξ and $\bar{\xi}$). It is found simultaneously with the optimal boundaries themselves /5/ as the solution of the unilateral curve of the problem

$$(a_0 - 2p_k)\,\omega_0(\eta) + 2\psi_0(\eta) = 0, \quad \eta \in L_k, \quad k = 1, 2, \ldots, m \tag{2.1}$$

generated by the **identity** (1.12) during the conformal mapping of the auxiliary plane of the variable η into the exterior of L_k, parallel slits or circles.

The absolute solvability of problem (1.14) for any values of the load parameters in the class of multivalued analytic functions is established in /4/. A natural requirement of single-valuedness imposes an additional constraint on the quantities a_0, b_0, which is derived as follows from the properties of the (single-valued) displacement vector $(u_0(z), v_0(z))$.

Let us subtract the component

$$\left(\frac{(\varkappa_0 - 1)a_0 - 2b_0}{8\mu_0} x, \; \frac{(\varkappa_0 - 1)a_0 + 2b_0}{8\mu_0} y\right)$$

corresponding to the homogeneous field (1.1) from it. The transformed vector $(u(z), v(z))$ decreases at infinity, by virtue of the identities $I_1(z) = \mathrm{const}$, $z \notin \Gamma$ is harmonic in the domains S_j, $j = 0, 1, \ldots, m$, is continuous everywhere, and according to (1.5), (1.6) and (1.3), takes the following values on Γ_k

$$u^+(\xi) = u^-(\xi) = (2\mu_0)^{-1}(Q_0 - p_k)\,x, \quad v^+(\xi) = v^-(\xi) = (2\mu_0)^{-1}(P_0 - p_k)\,y, \quad \xi = x + iy \in \Gamma_k, \quad k = 1, 2, \ldots, m \tag{2.2}$$

From the loading conditions it is found /1/ that

$$\frac{\partial u^+(\xi)}{\partial n} = -\frac{P_0 - p_k}{Q_0 - p_k}\frac{\partial u^-(\xi)}{\partial n}, \quad \frac{\partial v^+(\xi)}{\partial n} = \frac{Q_0 - p_k}{P_0 - p_k}\frac{\partial v^-(\xi)}{\partial n}, \; \xi \in \Gamma_k \tag{2.3}$$

Multiplying both sides of the first of the relationships (2.3) by $u(\xi)$, integrating over Γ_k and then summing over k we have

$$\int_{\Gamma_k} u^+(\xi)\frac{\partial u^+(\xi)}{\partial n} dl = \sum_{k=1}^{m} \frac{P_0 - p_k}{Q_0 - p_k}\int_{\Gamma_k} u^-(\xi)\frac{\partial u^-(\xi)}{\partial n} dl$$

According to Green's formula all the integrals constructed are non-negative, whence results the necessary condition for the existence of equally-strong boundaries (consideration of $v(z)$ in place of $u(z)$ results in it also)

$$\min_k (P_0 - p_k)(Q_0 - p_k)^{-1} \geqslant 0$$

(or taking (1.13) into account) into the equivalent form

$$\left|\frac{b_0}{a_0}\right| \leqslant \min_k \left|\frac{\mu_0(\varkappa_k - 1) - \mu_k(\varkappa_0 - 1)}{\mu_0(\varkappa_k - 1) + 2\mu_k}\right| \tag{2.4}$$

When all the $\mu_k = 0$ (a plane with equally-strong holes), inequality (2.4) goes over into the relationship $|b_0/a_0| \leqslant 1$ obtained /5/ as the requirement for schlichtness of the function $\omega_0(\eta)$ from (2.1). In the general case the allowable value $b_0 = \operatorname{dev} T_\infty$ has an upper limit set by the inclusion closest to the matrix in its elastic properties since the components of the tensor $T(z)$ should also be mutually close near the common boundary of such materials according to the conditions of continuity (1.5) and (1.6), and $\operatorname{dev} T(z) \equiv 0$ in conformity with (1.11) in all the inclusions.

Taking account of (2.4), the function $\omega_0(\eta)$ giving the outline of the optimal boundary is found /4/ from the numerical solution of the regular integral equation equivalent to problem (2.1). If the material of all the inclusions is identical, meaning, $p_k = p_1$, $k = 1, 2, \ldots, m$, then their optimal boundaries agree with the equally-strong boundaries of the free holes in a plane under the variable load $P_0' = P_0 + p_1$, $Q_0' = Q_0 + p_1$ at infinity, as follows from the preceding. For a certain symmetry in their arrangement problem (2.1) is solved by quadratures /5/. There are examples of such contours in /4-6/.

In particular, the equally-strong boundary of a single inclusion is an ellipse /5/ with $b_0/(a_0 - 2p_1)$ as the ratio of the axes. Homogeneity of the stress field within it is first noted without any relation to optimality in /7/.

3. Let the boundary Γ be optimal for the parameters P_0, Q_0 of the load (1.1) while the elastic moduli of all the inclusions are equal: $\mu_k = \mu_1$, $\nu_k = \nu_1$ so that also $\varkappa_k = \varkappa_1$. Then the solution of the boundary value problem (1.5) and (1.6) in closed form exists even in the more general case of arbitrary values of the forces at infinity $\sigma_x^\infty = P$, $\sigma_y^\infty = Q$, $\tau_{xy}^\infty = 0$. The potentials $\varphi_1(z)$ and $\psi_1(z)$ in the domain S_- are here linear in z while $\varphi_0(z)$ and $\psi_0(z)$ are expressed in terms of the function $\Omega_0(z)$ that is holomorphic in S_0 and decreases at infinity, from Sect.1 and which according to (1.12) and (1.13), satisfies the following identity on Γ

$$2\Omega_0(z) = d_0\bar{\xi} - b_0\xi, \quad d_0 = \frac{\mu_1(\varkappa_0 - 1) - \mu_0(\varkappa_1 - 1)}{2\mu_1 + \mu_0(\varkappa_1 - 1)} a_0, \quad \xi \in \Gamma \tag{3.1}$$

For the proof we represent the mentioned functions in a form that ensures their holomorphicity and the conservation by the necessary asymptotic form

$$\begin{aligned} &4\varphi_0(z) = az + 4D_0\Omega_0(z), \quad 2\psi_0(z) = bz + \\ &\quad 2D_0'\Omega_0(z) - 2[b_0z + 2\Omega_0(z)]\, d_0^{-1}D_0\Omega_0'(z) \\ &a = P + Q, \quad b = Q - P, \quad z \in S_0 \\ &\varphi_1(z) = D_1 z, \quad \psi_1(z) = D_1'z, \quad z \in S_- \end{aligned} \tag{3.2}$$

D_0, D_0', D_1, D_1' are certain real constants.

Substitution of these expressions into relationships (1.5) and (1.6) yields

$$\begin{aligned} &\mu_k\,[{}^1/_4(\varkappa_0 - 1)\,a\xi + \varkappa_0 D_0\Omega_0(\xi) - {}^1/_2 b\bar{\xi} - D_0'\overline{\Omega(\xi)}] - \\ &\quad \mu_0\,[(\varkappa_k - 1)\,D_1\xi - D_1'\bar{\xi}] = 0 \\ &{}^1/_2 a\xi + D_0\Omega_0(\xi) + {}^1/_2 b\bar{\xi} + D_0'\overline{\Omega_0(\xi)} - 2D_1\bar{\xi} - 2D_1'\bar{\xi} = 0 \end{aligned}$$

Replacing the function $\Omega_0(\xi)$ in the identities obtained by using condition (3.1) and then equating the coefficients of ξ and $\bar{\xi}$ to zero, we obtain a system of linear algebraic equations in the desired constants

$$\begin{aligned} &\mu_1\varkappa_0 b_0 D_0 + \mu_1 d_0 D_0' + 2\mu_0(\varkappa_1 - 1)\,D_1 = {}^1/_2\mu_1(\varkappa_0 - 1)\,a \\ &\mu_1\varkappa_0 d_0 D_0 + \mu_1 b_0 D_0' + 2\mu_0 D_1' = \mu_1 b \\ &b_0 D_0 - d_0 D_0' + 4D_1 = a, \quad -d_0 D_0 + b_0 D_0' + 2D_1' = b \end{aligned}$$

Its solution has the form

$$\begin{aligned} &D_0 = \Delta^{-1}(ab_0 - bd_0)(\mu_1 - \mu_0)\,[\mu_1(\varkappa_0 - 1) - \mu_0(\varkappa_1 - 1)], \\ &D_0' = \Delta^{-1}(b_0 b\Delta_1 - ad_0\Delta_2) \\ &4D_1 = a - b_0 D_0 + d_0 D_0', \quad 2D_1' = b + d_0 D_0 - b_0 D_0' \\ &\Delta = b_0^2\Delta_1 - a_0 d_0\Delta_2, \quad \Delta_1 = (\mu_1 - \mu_0)\,[2\mu_1\varkappa_0 - \mu_0(\varkappa_1 - 1)] \\ &\Delta_2 = (\mu_1\varkappa_0 + \mu_0)\,[\mu_1(\varkappa_0 - 1) - \mu_0(\varkappa_1 - 1)] \end{aligned}$$

It is seen that these constants are independent of the number and mutual location of the inclusions. When the load is optimal for a given shape of $\Gamma: P = P_0, Q = Q_0$, meaning

$a = a_0$, $b = b_0$, they understandably take the values found in Sect.1: $D_0 = D_1' = 0$, $D_0' = 1$, $2D_1 = p_1$.

It follows from (1.8) and (3.2) that the stress field in the inclusions with equally-strong boundaries remains homogeneous for any values of the load

$$\sigma_x(z) = 2D_1 + D_1', \quad \sigma_y(z) = 2D_1 - D_1', \ \tau_{xy}(z) = 0, \quad z \in S_-$$

although unlike (1.11) the tensor $T(z)$ indeed ceases to be, generally speaking, global. Therefore, the property of field uniformity is characteristic for classes of sets of plane curves of equally-strong shape parametrically dependent on the ratios b_0/a_0, $\varkappa_0/\varkappa_1$, μ_0/μ_1.

The stresses on Γ in the n, t coordinate system are found from (3.2) by means of the general formulas of plane elasticity theory /1/

$$\sigma_n^+(\xi) = \sigma_n^-(\xi) = 2D_1 + D_1'[1 - 2(\partial x/\partial n)^2]$$
$$\tau_{nt}^+(\xi) = \tau_{nt}^-(\xi) = 2D_1'\partial x/\partial n \times \partial y/\partial n$$
$$\sigma_t^+(\xi) = a - 4D_0b_0 - 2D_1 + (2D_0d_0 - D_1)[1 - 2(\partial x/\partial n)^2]$$
$$\sigma_t^-(\xi) = 2D_1 - D_1'[1 - 2(\partial x/\partial n)^2], \quad \xi = x + iy \in \Gamma$$

The easily verified relationships

$$\operatorname{Re}\frac{\partial\bar{\xi}}{\partial\xi} = 1 - 2\left(\frac{\partial x}{\partial n}\right)^2, \quad \operatorname{Im}\frac{\partial\bar{\xi}}{\partial\xi} = 2\frac{\partial x}{\partial n}\frac{\partial y}{\partial n}$$

as well as the identity (3.1) differentiated with respect to ξ are utilized here.

REFERENCES

1. LUR'YE A.I., Theory of Elasticity, Nauka, Moscow, 1970.
2. BANICHUK N.V., Optimization of the Shapes of Elastic Bodies. Nauka, Moscow, 1980.
3. KURSHIN L.M. and RASTORGUYEV G.I., The problem of the reinforcement of a hole contour in a plate by a flexible elastic rod, PMM, 44, 5, 1980.
4. VIGDERGAUZ S.B., On one case of the inverse problem of two-dimensional elasticity theory, PMM, 41, 5, 1977.
5. CHEREPANOV G.P., Inverse problems of plane elasticity theory, PMM, 38, 6, 1974.
6. VIGDERGAUZ S.B., An integral equation of the inverse problem of plane elasticity theory. PMM, 40, 3, 1976.
7. HARDIMAN N.J., Elliptic elastic inclusion in an infinite elastic plate, Quart. J. Mech. and Appl. Math. 7, Pt.2, 1954.

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